

Extremal Functions for the Singular Moser-Trudinger Inequality in 2 Dimensions

November 3, 2014

GYULA CSATÓ AND PROSENJIT ROY

Tata Institute of Fundamental Research, Centre For Applicable Mathematics, 560065 Bangalore, India
csato@math.tifrbng.res.in, prosenjit@math.tifrbng.res.in

Abstract

The Moser-Trudinger embedding has been generalized by Adimuthi and Sandeep to the following weighted version: if $\Omega \subset \mathbb{R}^2$ is bounded, $\alpha > 0$ and $\beta \in [0, 2)$ are such that

$$\frac{\alpha}{4\pi} + \frac{\beta}{2} \leq 1,$$

then

$$\sup_{\substack{v \in W_0^{1,2}(\Omega) \\ \|\nabla v\|_{L^2} \leq 1}} \int_{\Omega} \frac{e^{\alpha v^2} - 1}{|x|^{\beta}} \leq C.$$

We prove that the supremum is attained, generalizing a well-known result by Flucher, who has proved the case $\beta = 0$.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open smooth set. The Moser-Trudinger imbedding, which is due to Trudinger [12] and in its sharp form to Moser [10], states that the following supremum is finite

$$\sup_{\substack{v \in W_0^{1,2}(\Omega) \\ \|\nabla v\|_{L^2} \leq 1}} \int_{\Omega} (e^{4\pi v^2} - 1) < \infty.$$

First it has been shown by Carleson and Chang [3] that the supremum is actually attained, if Ω is a ball. In [11], Struwe proved that the result remains true if Ω

2010 Mathematics Subject Classification. Primary , Secondary .

Key words and phrases. Moser Trudinger embedding, extremal function.

is close to a ball in measure. Then Flucher [5] generalized this result to arbitrary domains in \mathbb{R}^2 . See also Malchiodi-Martinazzi [9] and the references therein for some recent developments on the subject.

The Moser-Trudinger embedding has been generalized by Adimurthi-Sandeep [1] to a singular version, which reads as the following: If $\alpha > 0$ and $\beta \in [0, 2)$ is such that

$$\frac{\alpha}{4\pi} + \frac{\beta}{2} \leq 1, \quad (1)$$

then the following supremum is finite

$$\sup_{\substack{v \in W_0^{1,2}(\Omega) \\ \|\nabla v\|_{L^2} \leq 1}} \int_{\Omega} \frac{e^{\alpha v^2} - 1}{|x|^{\beta}} < \infty.$$

We prove in this paper the following theorem, which states that the supremum is also attained for the singular Moser-Trudinger embedding.

Theorem 1 *Let Ω be a bounded open connected smooth subset of \mathbb{R}^2 , $\alpha > 0$ and $\beta \in [0, 2)$ be such that (1) is satisfied. Then there exists $u \in W_0^{1,2}(\Omega)$ such that $\|\nabla u\|_{L^2} \leq 1$ and*

$$\sup_{\substack{v \in W_0^{1,2}(\Omega) \\ \|\nabla v\|_{L^2} \leq 1}} \int_{\Omega} \frac{e^{\alpha v^2} - 1}{|x|^{\beta}} = \int_{\Omega} \frac{e^{\alpha u^2} - 1}{|x|^{\beta}}.$$

The essential difficulty is that the functional

$$u \in W_0^{1,2}(\Omega) \rightarrow F_{\Omega}(u) = \int_{\Omega} \frac{e^{\alpha u^2} - 1}{|x|^{\beta}}$$

is not continuous with respect to weak convergence. To see this fact one can take the usual Moser-sequence, as in Flucher [5] page 472. The proof of this theorem follows the ideas of Flucher and is based on a concentration compactness alternative by Lions [8]. We give here an outline of the proof, explaining what is new compared to Flucher's result and what is not. The proof is divided into 6 sections (Section 1 is introduction).

Section 2. We introduce some notation and definitions and recall some of their properties.

Section 3. The concentration compactness alternative (see Theorem 6) applies to the new functional F_{Ω} without change and states that if a sequence u_i does not concentrate at any point of Ω , then up to a subsequence $\lim_{i \rightarrow \infty} F_{\Omega}(u_i) = F_{\Omega}(u)$. We prove in this section that the hypothesis for the concentration compactness alternative is satisfied for the new singular functional F_{Ω} . This is essentially the same as for

the Moser-Trudinger functional. However, we show that for the functional F_Ω it is sufficient to consider the case when $0 \in \overline{\Omega}$ and a maximizing sequence concentrates at 0 (see Proposition 7).

Section 4. We show that the supremum is attained if Ω is the ball by using the result of Carleson-Chang [3] (see Theorem 12) and the transformation introduced by Adimurthi-Sandeep [1], which relates F_Ω to the classical Moser-Trudinger functional for radial functions (see Lemma 10). We will in particular deduce the following strict inequality (see Theorem 15)

$$F_{B_1}^\delta(0) < F_{B_1}^{\sup}, \quad (2)$$

where $F_{B_1}^\delta(0)$ denotes the concentration level at 0 and the right hand side denotes the supremum of F_{B_1} .

Section 5. In this section we establish the inequality

$$F_\Omega^{\sup} \geq I_\Omega(0)^{2-\beta} F_{B_1}^{\sup}, \quad (3)$$

where $I_\Omega(0)$ is the conformal incenter of Ω at 0 (see Theorem 16). It consists of constructing for any given radial function v on the ball a corresponding function given on Ω , which satisfies the estimate $F_\Omega(u) \geq I_\Omega(0)^{2-\beta} F_{B_1}(v)$. In this step there is a crucial difference with Flucher's result where the inequality is deduced from the isoperimetric inequality. To carry out the same construction we needed a singularly weighted isoperimetric inequality, which is on its own a deep result with many consequences. It has been established in a separate paper in Csató [4].

Section 6. In this section we prove a reverse inequality to (3) for concentrating sequences: given a concentrating sequence $\{u_i\}$ at 0 which maximizes the concentration level $F_\Omega^\delta(0)$, one can construct a sequence v_i such that

$$F_\Omega^\delta(0) = \lim_{i \rightarrow \infty} F_\Omega^\delta(u_i) \leq I_\Omega^{2-\beta}(0) \liminf_{i \rightarrow \infty} F_{B_1}(v_i) \leq I_\Omega^{2-\beta}(0) F_{B_1}^\delta(0), \quad (4)$$

see Proposition 22. This will imply the concentration formula

$$F_\Omega^\delta(0) = I_\Omega^{2-\beta}(0) F_{B_1}^\delta(0), \quad (5)$$

see Theorem 21. An essential difference is that if $\beta = 0$, then $F_\Omega^\delta(x) = I_\Omega^2(x) F_{B_1}^\delta(0)$ for all $x \in \Omega$, see Flucher [5], which implies, a priori, that a maximizing sequence can concentrate only at the point x where $I_\Omega(x)$ is maximal. Clearly $I_\Omega(x)$ is independent of the functional. This is in strong contrast to our case ($\beta > 0$), where (5) holds only at zero, since $F_\Omega^\delta(x) = 0$ if $x \neq 0$. This is due to the dependence in x of the integrand of the functional F_Ω . In particular, the map $x \mapsto F_\Omega^\delta(x)$ is not continuous, unless $\beta = 0$. The proof of (4) is long and technical. We made a great effort to give clear and rigorous proofs of all steps and our presentation differs significantly from Flucher's paper (see also Remarks 28 and 30).

Section 7. We prove here Theorem 1. Combining the inequalities (5), (2) and then (3), one obtains that

$$F_{\Omega}^{\delta}(0) < F_{\Omega}^{\sup}.$$

One deduces from this strict inequality that a maximizing sequence cannot concentrate at 0. It now follows easily from the results of Section 3 that the maximum is attained.

2 Notations and Definitions

Throughout this paper $\Omega \subset \mathbb{R}^2$ will denote a bounded open set with smooth boundary $\partial\Omega$. Its 2-dimensional area is written as $|\Omega|$. The 1-dimensional Hausdorff measure is denoted by σ . Balls with radius R and center at x are written $B_R(x) \subset \mathbb{R}^2$; if $x = 0$, we simply write B_R . The space $W^{1,2}(\Omega)$ denotes the usual Sobolev space of functions and $W_0^{1,2}(\Omega)$ those Sobolev functions with vanishing trace on the boundary. Throughout this paper $\alpha, \beta \in \mathbb{R}$ are two constants satisfying $\alpha > 0$, $\beta \in [0, 2)$ and

$$\frac{\alpha}{4\pi} + \frac{\beta}{2} \leq 1.$$

– We define the the funtctional $F_{\Omega}, J_{\Omega} : W_0^{1,2}(\Omega) \rightarrow \mathbb{R}$ by

$$\begin{aligned} F_{\Omega}(u) &= \int_{\Omega} \frac{e^{\alpha u^2} - 1}{|x|^{\beta}} dx, \\ J_{\Omega}(u) &= \int_{\Omega} \left(e^{4\pi u^2} - 1 \right) dx. \end{aligned}$$

– We say that a sequence $\{u_i\} \subset W_0^{1,2}(\Omega)$ concentrates at $x \in \overline{\Omega}$ if

$$\lim_{i \rightarrow \infty} \|\nabla u_i\|_{L^2} = 1 \quad \text{and} \quad \forall \epsilon > 0 \quad \lim_{i \rightarrow \infty} \int_{\Omega \setminus B_{\epsilon}(x)} |\nabla u_i|^2 = 0.$$

This definition implies the convergence $|\nabla u_i|^2 dx \rightharpoonup \delta_x$ weakly in meausre, where δ_x is the Dirac measure at x . We will use the following well known property of concentrating sequences: if $\{u_i\}$ concentrates, then $u_i \rightharpoonup 0$ in $W^{1,2}(\Omega)$. In particular

$$u_i \rightarrow 0 \quad \text{in } L^2(\Omega), \tag{6}$$

see for instance Flucher [5] Step 1 page 478.

– We define the sets

$$W_{0,rad}^{1,2}(B_1) = \left\{ u \in W_0^{1,2}(B_1) \mid u \text{ is radial} \right\}$$

and analogously $C_{c,rad}^\infty(B_1)$ is the set of radially symmetric smooth functions with compact support in B_1 . By abuse of notation we will usually write $u(x) = u(|x|)$ for $u \in W_{0,rad}^{1,2}(B_1)$. Recall that $C_{c,rad}^\infty(B_1)$ is dense in $W_{0,rad}^{1,2}(B_1)$ in the $W^{1,2}$ norm. If in addition u is radially decreasing we write $u \in W_{0,rad\searrow}^{1,2}(B_1)$, respectively $u \in C_{c,rad\searrow}^\infty(B_1)$.

– Define

$$\mathcal{B}_1(\Omega) = \{u \in W_0^{1,2}(\Omega) \mid \|\nabla u\|_{L^2} \leq 1\}.$$

– Finally we define

$$F_\Omega^{\sup} = \sup_{u \in \mathcal{B}_1(\Omega)} F_\Omega(u).$$

J_Ω^{\sup} is defined in an analogous way, replacing F by J . If $x \in \overline{\Omega}$ and the supremum is taken only over concentrating sequences, we write $F_\Omega^\delta(x)$, more precisely

$$F_\Omega^\delta(x) = \sup \left\{ \limsup_{i \rightarrow \infty} F_\Omega(u_i) \mid \{u_i\} \subset \mathcal{B}_1(\Omega) \text{ concentrates at } x \right\}.$$

We define in an analogous way $J_\Omega^\delta(x)$. If $\Omega = B_1$, then we define

$$F_{B_1,rad\searrow}^{\sup} = \sup_{u \in W_{0,rad\searrow}^{1,2}(B_1) \cap \mathcal{B}_1(B_1)} F_{B_1}(u),$$

$$F_{B_1,rad\searrow}^\delta(0) = \sup \left\{ \limsup_{i \rightarrow \infty} F_{B_1}(u_i) \mid \{u_i\} \subset W_{0,rad\searrow}^{1,2}(B_1) \cap \mathcal{B}_1(B_1) \text{ concentrates at } 0 \right\}.$$

We define $J_{B_1,rad\searrow}^{\sup}$ and $J_{B_1,rad\searrow}^\delta(0)$ in an analogous way.

– If $\Omega \subset \mathbb{R}^2$ then Ω^* is its symmetric rearrangement, that is $\Omega^* = B_R(0)$, where $|\Omega| = \pi R^2$. If $u \in W_0^{1,2}(\Omega)$, then $u^* \in W_{0,rad\searrow}^{1,2}(B_R(0))$ will denote the Schwarz symmetrization of u . For basic properties of the Schwarz symmetrization we refer to Kesavan [7], Chapters 1 and 2, which we will use throughout. In particular we will use frequently and without further comment that if $u \in W_0^{1,2}(\Omega)$, then u^* satisfies

$$F_\Omega(u) \leq F_{B_R}(u^*) \quad \text{and} \quad \|\nabla u^*\|_{L^2(B_R)} \leq \|\nabla u\|_{L^2(\Omega)}.$$

We will additionally need, as in Flucher [5], a slight modification of the Hardy-Littlewood, respectively Pólya-Szegő theorem, stated in the next proposition.

Proposition 2 (i) *Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, where $1/p + 1/q = 1$. Then for any $a \in \mathbb{R}$*

$$\int_{\{f \geq a\}} f g \leq \int_{\{f^* \geq a\}} f^* g^*.$$

(ii) Let $u \in W_0^{1,2}(\Omega)$ such that $u \geq 0$. Then for any $t \in (0, \infty)$

$$\int_{\{u^* \leq t\}} |\nabla u^*|^2 \leq \int_{\{u \leq t\}} |\nabla u|^2 \quad \text{and} \quad \int_{\{u^* \geq t\}} |\nabla u^*|^2 \leq \int_{\{u \geq t\}} |\nabla u|^2.$$

– We say that a sequence of sets $\{A_i\} \subset \mathbb{R}^2$ are approximately small disks at $x \in \mathbb{R}^2$ (of radius τ_i) as $i \rightarrow \infty$ if there exists sequences $\tau_i, \sigma_i > 0$ such that $\lim_{i \rightarrow \infty} \tau_i = 0$,

$$\lim_{i \rightarrow \infty} \frac{\sigma_i}{\tau_i} = 0$$

and

$$B_{\tau_i - \sigma_i}(x) \subset A_i \subset B_{\tau_i + \sigma_i}(x) \quad \text{for all } i \text{ big enough.}$$

– If $x \in \Omega$, then $G_{\Omega,x}$ will denote the Green's function of Ω with singularity at x . It can always be decomposed in the form

$$G_{\Omega,x}(y) = -\frac{1}{2\pi} \log(|x - y|) - H_{\Omega,x}(y), \quad y \in \Omega \setminus \{x\},$$

where H is the regular part. The conformal incenter $I_\Omega(x)$ of Ω at x is defined by

$$I_\Omega(x) = e^{-2\pi H_{\Omega,x}(x)}.$$

We refer to Flucher [5] concerning properties and examples regarding the conformal incenter, cf. [5] Lemma 10 and Proposition 12 (see also [4] Lemma 12). We will need in particular the following results:

Proposition 3 *Let $x \in \Omega$. Then $G_{\Omega,x}$ and $I_\Omega(x)$ have the following properties:*

(a) *For every $t \in [0, \infty)$*

$$\int_{\{G_{\Omega,x} < t\}} |\nabla G_{\Omega,x}(y)|^2 dy = t.$$

(b) *For every $t \in [0, \infty)$*

$$\int_{\{G_{\Omega,x} = t\}} |\nabla G_{\Omega,x}(y)| d\sigma = 1.$$

(c)

$$\lim_{t \rightarrow \infty} \frac{|\{G_{\Omega,x} > t\}|}{e^{-4\pi t}} = \pi (I_\Omega(x))^2.$$

(d) *If $B_R = \Omega^*$ is the symmetrized domain, then*

$$I_\Omega(x) \leq I_{B_R}(0) = R.$$

(e) *If $t_i \geq 0$ is a given sequence such that $t_i \rightarrow \infty$, then the sets $\{G_{\Omega,x} > t_i\}$ are approximately small disks at x of radius $\tau_i = I_\Omega(x)e^{-2\pi t_i}$.*

3 Some Preliminary Results

We first note that it is sufficient to work with non-negative smooth maximizing sequences. More precisely we have the following lemma, which we will use in Section 6 in a crucial way.

Lemma 4 *Let $\{u_i\} \subset \mathcal{B}_1(\Omega)$ be a sequence such that the limit $\lim_{i \rightarrow \infty} F_\Omega(u_i)$ exists. Then there exists a sequence $\{w_i\} \subset \mathcal{B}_1(\Omega) \cap C_c^\infty(\Omega)$ such that*

$$\liminf_{i \rightarrow \infty} F_\Omega(w_i) \geq \lim_{i \rightarrow \infty} F_\Omega(u_i).$$

Moreover, if u_i concentrates at $x_0 \in \overline{\Omega}$, then also w_i concentrates at x_0 . In particular maximizing sequences for F_Ω^{\sup} and $F_\Omega^\delta(x_0)$ can always be assumed to be smooth and non-negative.

Proof For each $i \in \mathbb{N}$ there exists $\{v_i^k\} \subset C_c^\infty(\Omega)$ such that $v_i^k \rightarrow u_i$ almost everywhere in Ω ,

$$v_i^k \rightarrow u_i \text{ in } W^{1,2}(\Omega) \text{ for } k \rightarrow \infty \quad \text{and} \quad \|\nabla v_i^k\|_{L^2} = \|\nabla u_i\|_{L^2} \text{ for all } k.$$

Using Fatou's lemma there exists $k_1(i) \in \mathbb{N}$ such that

$$F_\Omega(u_i) \leq F_\Omega(v_i^j) + \frac{1}{2^i} \quad \forall j \geq k_1(i).$$

Moreover, from the convergence in $W^{1,2}(\Omega)$ we also obtain the existence of $k_2(i) \in \mathbb{N}$, such that

$$\|\nabla v_i^j - \nabla u_i\|_{L^2(\Omega)} \leq \frac{1}{2^i} \quad \forall j \geq k_2(i).$$

We now define $w_i = v_i^{k(i)}$, where $k(i) = \max\{k_1(i), k_2(i)\}$. It can be easily verified that w_i has all the desired properties. ■

Lemma 5 (compactness in interior) *Let $0 < \eta < 1$ and suppose $\{u_i\} \subset W_0^{1,2}(\Omega)$ is such that*

$$\limsup_{i \rightarrow \infty} \|\nabla u_i\|_{L^2} \leq \eta \quad \text{and} \quad u_i \rightharpoonup u \text{ in } W^{1,2}(\Omega)$$

for some $u \in W^{1,2}(\Omega)$. Then for some subsequence

$$\frac{e^{\alpha u_i^2}}{|x|^\beta} \rightarrow \frac{e^{\alpha u^2}}{|x|^\beta} \quad \text{in } L^1(\Omega)$$

and in particular $\lim_{i \rightarrow \infty} F_\Omega(u_i) = F_\Omega(u)$.

Proof The idea of the proof is to apply Vitali convergence theorem. We can assume that, up to a subsequence, that $u_i \rightarrow u$ almost everywhere in Ω and that

$$\|\nabla u_i\|_{L^2} \leq \theta = \frac{1+\eta}{2} < 1 \quad \forall i \in \mathbb{N}.$$

We can therefore define $v_i = u_i/\theta \in \mathcal{B}_1(\Omega)$, which satisfies $\|\nabla v_i\|_{L^2} \leq 1$ for all i . Moreover let us define $\bar{\alpha} = \alpha\theta^2 < \alpha$, such that

$$\frac{\bar{\alpha}}{4\pi} + \frac{\beta}{2} < 1.$$

Let $E \subset \Omega$ be an arbitrary measurable set. We use Hölder inequality with exponents r and s , where

$$r = \frac{4\pi}{\bar{\alpha}} > 1 \quad \text{and} \quad \frac{1}{s} = 1 - \frac{1}{r} > \frac{\beta}{2},$$

to obtain that

$$\int_E \frac{e^{\alpha u_i^2}}{|x|^\beta} = \int_E \frac{e^{\bar{\alpha} v_i^2}}{|x|^\beta} \leq \left(\int_E e^{4\pi v_i^2} \right)^{\frac{1}{r}} \left(\int_E \frac{1}{|x|^{\beta s}} \right)^{\frac{1}{s}}.$$

Let $\epsilon > 0$ be given. In view of the Moser-Trudinger inequality and using that $1/|x|^{\beta s} \in L^1(\Omega)$, we obtain that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\int_E \frac{e^{\alpha u_i^2}}{|x|^\beta} \leq \epsilon \quad \forall |E| \leq \delta \text{ and } i \in \mathbb{N}.$$

This shows that the sequence $e^{\alpha u_i^2}/|x|^\beta$ is equi-integrable and the Vitali convergence theorem yields convergence in $L^1(\Omega)$. This proves the lemma. ■

Theorem 6 (Concentration-Compactness Alternative) *Let $\{u_i\} \subset \mathcal{B}_1(\Omega)$. Then there is a subsequence and $u \in W_0^{1,2}(\Omega)$ with $u_i \rightharpoonup u$ in $W^{1,2}(\Omega)$, such that either*

(a) $\{u_i\}$ concentrates at a point $x \in \overline{\Omega}$,

or

(b) the following convergence holds true

$$\lim_{i \rightarrow \infty} F_\Omega(u_i) = F_\Omega(u).$$

Proof This is a direct application of Theorem 1 in Flucher [5]. Lemma 5 shows precisely that the hypothesis of F_Ω being compact in the interior is satisfied. ■

Proposition 7 *Let $\beta > 0$, $\{u_i\} \subset \mathcal{B}_1(\Omega)$ and suppose that u_i concentrates at $x_0 \in \overline{\Omega}$, where $x_0 \neq 0$. Then one has that, for some subsequence, $u_i \rightharpoonup 0$ in $W^{1,2}(\Omega)$ and*

$$\lim_{i \rightarrow \infty} F_\Omega(u_i) = F_\Omega(0) = 0.$$

In particular $F_\Omega^\delta(x_0) = 0$.

Remark 8 Note that $F_\Omega^\delta(0) > 0$. This follows by rescaling the Moser sequence (see Flucher [5] page 472) on a small ball around the origin and extending it by 0 in Ω .

Proof Since $x_0 \neq 0$, there exists an $\epsilon > 0$ such that $B_\epsilon(x_0) \cap B_{2\epsilon}(0) = \emptyset$. We extend u_i by 0 in $\mathbb{R}^2 \setminus \overline{\Omega}$ and split the integral in the following way

$$F_\Omega(u_i) = \int_{\Omega \setminus B_\epsilon(0)} \frac{e^{\alpha u_i^2} - 1}{|x|^\beta} + \int_{B_\epsilon(0)} \frac{e^{\alpha u_i^2} - 1}{|x|^\beta} = A_i + B_i,$$

A_i can obviously be estimated by

$$A_i \leq \frac{1}{|\epsilon|^\beta} \int_{\Omega} \left(e^{\alpha u_i^2} - 1 \right).$$

Since $\beta > 0$, it follows that $\alpha < 4\pi$, and it follows easily from (6) and from Vitali convergence theorem (similarly as in the proof of Lemma 5) that $\lim_{i \rightarrow \infty} A_i = 0$. We now show that $\lim_{i \rightarrow \infty} B_i = 0$. Choose $\eta \in C_c^\infty(\Omega)$ such that $\eta \geq 0$ and

$$\eta = 1 \quad \text{in } B_\epsilon(0), \quad \eta = 0 \quad \text{in } \Omega \setminus B_{2\epsilon}(0).$$

Define $w_i = \eta u_i \in W_0^{1,2}(B_{2\epsilon}(0))$ and note that

$$\begin{aligned} \int_{B_{2\epsilon}(0)} |\nabla(\eta u_i)|^2 &\leq 2 \int_{B_{2\epsilon}(0)} \eta^2 |\nabla u_i|^2 + 2 \int_{B_{2\epsilon}(0)} |\nabla \eta|^2 |u_i|^2 \\ &\leq 2 \int_{\Omega \setminus B_\epsilon(x_0)} |\nabla u_i|^2 + C_\eta \int_{\Omega} |u_i|^2. \end{aligned}$$

Since $\{u_i\}$ concentrates at 0, we get that for some $i_0 \in \mathbb{N}$

$$\int_{B_{2\epsilon}(0)} |\nabla(\eta u_i)|^2 \leq \frac{1}{2} \quad \forall i \geq i_0.$$

We can therefore apply Lemma 5 to the sequence $\{\eta u_i\}$ and the domain $B_{2\epsilon}(0)$ to get that

$$0 \leq \lim_{i \rightarrow \infty} B_i \leq \lim_{i \rightarrow \infty} \int_{B_{2\epsilon}(0)} \frac{e^{\alpha(\eta u_i)^2} - 1}{|x|^\beta} = 0,$$

where we have used again (6). ■

We first prove Theorem 1 for some simple cases, which is the content of the next proposition.

Proposition 9 *There exists $u \in \mathcal{B}_1(\Omega)$ such that $F_\Omega(u) = F_\Omega^{\sup}$ in the following cases:*

$$(i) \quad 0 \notin \overline{\Omega} \quad \text{or} \quad (ii) \quad \frac{\alpha}{4\pi} + \frac{\beta}{2} < 1.$$

Proof Let $\{u_i\} \subset \mathcal{B}_1(\Omega)$ be a maximizing sequence, that is

$$F_\Omega^{\sup} = \lim_{i \rightarrow \infty} F_\Omega(u_i). \quad (7)$$

We can assume that $u_i \rightharpoonup u \in \mathcal{B}_1(\Omega)$ in $W^{1,2}(\Omega)$.

Part (i). We can assume, in view of Flucher's result [5], that $\beta > 0$, which implies that $\alpha < 4\pi$. Since $0 \notin \overline{\Omega}$, there exists a constant C_1 such that $1/|x|^\beta \leq C_1$. Therefore we can proceed similarly as in the proof of Lemma 5, or Proposition 7, by using Hölder inequality and Vitali convergence theorem and obtain that

$$\lim_{i \rightarrow \infty} F_\Omega(u_i) = F_\Omega(u).$$

Part (ii). Let $\gamma > 1$ be such that

$$\gamma^2 \frac{\alpha}{4\pi} = 1 - \frac{\beta}{2}.$$

We set $\bar{\alpha} = \gamma^2 \alpha$ which satisfies $\bar{\alpha}/(4\pi) + \beta/2 = 1$. We define $v_i = u_i/\gamma$, which satisfies $v_i \rightharpoonup v := u/\gamma$ in $W^{1,2}(\Omega)$ and

$$\|\nabla v_i\|_{L^2} \leq \frac{1}{\gamma} < 1 \quad \forall i.$$

We therefore get from Lemma 5 that

$$\lim_{i \rightarrow \infty} \int_\Omega \frac{e^{\alpha u_i^2}}{|x|^\beta} = \lim_{i \rightarrow \infty} \int_\Omega \frac{e^{\bar{\alpha} v_i^2}}{|x|^\beta} = \int_\Omega \frac{e^{\bar{\alpha} v^2}}{|x|^\beta} = \int_\Omega \frac{e^{\alpha u^2}}{|x|^\beta},$$

from which the statement of Part (ii) follows. ■

4 The Case $\Omega = B_1$.

In this section we deal with the case where Ω is the unit ball. The following lemma is essentially due to Adimurthi-Sandeep [1].

Lemma 10 *Let $0 < a < \infty$, and u be radial function on B_1 . Define*

$$T_a(u)(x) = \sqrt{a}u \left(|x|^{\frac{1}{a}} \right).$$

Then T_a satisfies that

$$T_a : W_{0,rad}^{1,2}(B_1) \rightarrow W_{0,rad}^{1,2}(B_1).$$

T_a is invertible with $(T_a)^{-1} = T_{1/a}$ and it satisfies

$$\|\nabla(T_a(u))\|_{L^2} = \|\nabla u\|_{L^2} \quad \forall u \in W_{0,rad}^{1,2}(B_1).$$

Moreover if $a = 1 - \frac{\beta}{2}$, then

$$F_{B_1}(u) = \frac{1}{a} J_{B_1}(T_a(u)) + \frac{|B_1|}{a} - \int_{B_1} \frac{1}{|x|^\beta} \quad \forall u \in W_{0,rad}^{1,2}(B_1). \quad (8)$$

Proof Step 1. Let us first show that T_a satisfies

$$T_a(C_{c,rad}^\infty(B_1)) \subset W_{0,rad}^{1,2}(B_1).$$

and

$$\int_{B_1} |\nabla(T_a(u))|^2 = \int_{B_1} |\nabla u|^2. \quad (9)$$

Let $u \in C_{c,rad}^\infty(B_1)$ and $v = T_a(u)$. Since u is bounded, so is also v . In particular we immediately obtain that $v \in L^2(B_1)$. Note that in general $v \notin C^1(B_1)$ (take for example $a < 1$). However we will show that v has weak derivatives. First note that ∇v is well defined on $B_1 \setminus \{0\}$, since $v \in C^1(B_1 \setminus \{0\})$. Moreover we have, by the change of variable $r = s^a$ that

$$\int_{B_1} |\nabla v|^2 = \frac{1}{a} \int_0^1 \left(u' \left(r^{\frac{1}{a}}\right)\right)^2 r^{\frac{2}{a}-2} 2\pi r dr = \int_0^1 (u'(s))^2 2\pi s ds \quad (10)$$

This proves (9). Since $v \in C^1(B_1 \setminus \{0\})$, we have that for any $\epsilon > 0$,

$$\int_{B_1 \setminus B_\epsilon(0)} v \frac{\partial \varphi}{\partial x_i} = - \int_{B_1 \setminus B_\epsilon(0)} \frac{\partial v}{\partial x_i} \varphi + \int_{\partial B_\epsilon(0)} v \varphi \nu_i \quad \forall \varphi \in C_c^\infty(B_1),$$

where $\nu = (\nu_1, \nu_2)$ is the unit outward normal on ∂B_1 . In view of (9) we have that $\|\nabla v\|_{L^2} < \infty$. Therefore, recalling also $\|v\|_{L^\infty} \leq \infty$, we obtain by letting $\epsilon \rightarrow 0$, that

$$\int_{B_1} v \frac{\partial \varphi}{\partial x_i} = - \int_{B_1} \frac{\partial v}{\partial x_i} \varphi \quad \forall \varphi \in C_c^\infty(B_1).$$

This shows that $v \in W_0^{1,2}(B_1)$. We can therefore apply Pioncaré inequality to obtain a constant C which satisfies

$$\|T_a(u)\|_{W^{1,2}} \leq C \|\nabla u\|_{L^2} \quad \forall u \in C_{c,rad}^\infty(B_1).$$

We can now extend by a density argument T_a to an operator defined on whole $W_{0,rad}^{1,2}(B_1)$.

Step 2. It remains to show (8). Let again v be given by $v = T_a(u)$. Recall that the assumptions on a and β imply that $a = \frac{\alpha}{4\pi}$. Thus, by using the substitution $r = s^{1/a}$, we get

$$\begin{aligned} F_{B_1}(u) + \int_{B_1} \frac{1}{|x|^\beta} &= \int_0^1 \frac{e^{\alpha u(r)^2}}{r^\beta} 2\pi r dr = \frac{1}{a} \int_0^1 e^{4\pi(\sqrt{a}u(s^{\frac{1}{a}}))^2} ds \\ &= \frac{1}{a} J_{B_1}(v) + \frac{1}{a} \int_{B_1} 1 = \frac{1}{a} J_{B_1}(v) + \frac{|B_1|}{a}. \end{aligned}$$

This concludes the proof of the last statement. ■

The following corollary follows easily from Lemma 10.

Corollary 11 *Let $a = 1 - \beta/2$. Then the following identities hold true*

$$\sup_{u \in W_{0,rad}^{1,2}(B_1) \cap \mathcal{B}_1(B_1)} F_{B_1}(u) = \frac{1}{a} \sup_{u \in W_{0,rad}^{1,2}(B_1) \cap \mathcal{B}_1(B_1)} J_{B_1}(u) + \frac{|B_1|}{a} - \int_{B_1} \frac{1}{|x|^\beta},$$

and

$$F_{B_1}^{sup} = \frac{1}{a} J_{B_1}^{sup} + \frac{|B_1|}{a} - \int_{B_1} \frac{1}{|x|^\beta}.$$

Proof The first equality follows directly from Lemma 10. By Schwarz symmetrization, the two equalities of the corollary are equivalent. ■

One of the crucial ingredients of the proof is the following result of Carleson and Chang [3]. Essential is the strict inequality in the following theorem. The second equality is an immediate consequence of the properties of Schwarz symmetrization.

Theorem 12 (Carleson-Chang) *The following strict inequality holds true*

$$J_{B_1, rad \searrow}^\delta(0) < J_{B_1, rad \searrow}^{sup} = J_{B_1}^{sup}.$$

Remark 13 The result in Carleson and Chang is acutally more precise, stating that

$$\pi e = \sup_{x \in \overline{B_1}} J_{B_1, rad \searrow}^\delta(x) < J_{B_1, rad \searrow}^{sup},$$

but for our purpose we only need an estimate for the concentration level at 0.

From Lemma 10 and Theorem 12 we easily deduce the following proposition.

Lemma 14 *Let $\{u_i\} \subset \mathcal{B}_1(B_1)$ be a sequence which concentrates at 0. If $\{u_i^*\}$ also concentrates at 0, then the following strict inequality holds true*

$$\limsup_{i \rightarrow \infty} F_{B_1}(u_i) < F_{B_1}^{sup}.$$

Proof Let $a = 1 - \beta/2$ and define $v_i = T_a(u_i^*)$. Let us first show that $\{v_i\}$ concentrates at 0. From Lemma 10 we know that $\lim_{i \rightarrow \infty} \|\nabla v_i\|_{L^2} = \lim_{i \rightarrow \infty} \|\nabla u_i^*\|_{L^2} = 1$. Moreover, the same calculation which shows this identity, namely (10), shows also that for any $\epsilon > 0$

$$\int_{B_1 \setminus B_\epsilon(0)} |\nabla v_i|^2 = \int_{B_1 \setminus B_{\frac{1}{\epsilon a}}(0)} |\nabla u_i^*|^2.$$

It now follows that v_i concentrates at 0, since u_i^* does. From the properties of the symmetric rearrangement, namely $F_{B_1}(u_i) \leq F_{B_1}(u_i^*)$, and from Lemma 10 we obtain that

$$\begin{aligned} \limsup_{i \rightarrow \infty} F_{B_1}(u_i) &\leq \limsup_{i \rightarrow \infty} F_{B_1}(u_i^*) = \frac{1}{a} \limsup_{i \rightarrow \infty} J_{B_1}(v_i) + \frac{|B_1|}{a} - \int_{B_1} \frac{1}{|x|^\beta} \\ &\leq \frac{1}{a} J_{B_1, rad \searrow}^\delta(0) + \frac{|B_1|}{a} - \int_{B_1} \frac{1}{|x|^\beta}. \end{aligned}$$

We now apply Theorem 12 and Corollary 11 to obtain that

$$\limsup_{i \rightarrow \infty} F_{B_1}(u_i) < \frac{1}{a} J_{B_1}^{\sup} + \frac{|B_1|}{a} - \int_{B_1} \frac{1}{|x|^\beta} = F_{B_1}^{\sup},$$

which proves the proposition ■

A consequence of Lemma 14 is the following theorem, stating that the supremum of F_{B_1} is attained.

Theorem 15 *The following strict inequality holds*

$$F_{B_1}^\delta(0) < F_{B_1}^{\sup}.$$

In particular there exists $u \in \mathcal{B}_1(B_1)$ such that $F_{B_1}^{\sup} = F_{B_1}(u)$.

Proof Let $\{u_i\} \subset \mathcal{B}_1(B_1)$ be a concentrating sequence at 0, which maximizes $F_{B_1}^\delta(0)$. Using (6) and the properties of symmetrization we obtain that $u_i^* \rightarrow 0$ in $L^2(B_1)$. Therefore u_i^* must concentrate, otherwise we get from Theorem 6 and Remark 8 the contradiction (taking again some subsequence)

$$0 < F_{B_1}^\delta(0) = \lim_{i \rightarrow \infty} F_{B_1}(u_i) \leq \lim_{i \rightarrow \infty} F_{B_1}(u_i^*) = F_{B_1}(0) = 0.$$

Thus u_i^* must concentrate at 0. We can apply lemma 14 to obtain that

$$F_{B_1}^\delta(0) < F_{B_1}^{\sup}.$$

The second statement of the theorem follows from this strict inequality, Theorem 6 and Proposition 7. ■

5 Ball to Domain Construction

In view of Proposition 9, it remains to prove Theorem 1 for general domain with $0 \in \overline{\Omega}$, when $\alpha/(4\pi) + \beta/2 = 1$, and we can also take $\beta > 0$. Hence from now on we always assume that we are in this case. In addition, we assume in this section and Section 6 that $0 \in \Omega$. The ball to domain construction is given by the following definition: for $v \in W_{0,rad}^{1,2}(B_1)$ and $x \in \Omega$, define $P_x(v) = u : \Omega \setminus \{x\} \rightarrow \mathbb{R}$ by

$$P_x(v)(y) = v \left(e^{-2\pi G_{\Omega,x}(y)} \right) = v \left((G_{B_1,0})^{-1} (G_{\Omega,x}(y)) \right),$$

where, by abuse of notation, we have identified v and $G_{B_1,0}$ with the corresponding radial function. The main result of this section is the following theorem.

Theorem 16 *For any $v \in W_{0,rad}^{1,2}(B_1) \cap \mathcal{B}_1(B_1)$ define $u = P_0(v)$. Then $u \in \mathcal{B}_1(\Omega)$ and it satisfies*

$$F_{\Omega}(u) \geq I_{\Omega}(0)^{2-\beta} F_{B_1}(v).$$

In particular the following inequality holds true

$$F_{\Omega}^{sup} \geq I_{\Omega}(0)^{2-\beta} F_{B_1}^{sup}.$$

Moreover if $\{v_i\} \subset W_{0,rad}^{1,2}(B_1)$ concentrates at 0, then $u_i = P_0(v_i)$ concentrates at 0.

The proof of this theorem is based on the following theorem proven in Csátó [4] Theorem 12, which is a consequence of a weighted isoperimetric inequality.

Theorem 17 *Let Ω be a bounded open smooth connected set with $0 \in \Omega$ and let also $x \in \Omega$. then the following inequality holds true*

$$|\Omega|^{1-\frac{\beta}{2}} \leq \frac{1}{4\pi^{1+\frac{\beta}{2}}} \int_{\partial\Omega} \frac{1}{|y|^{\beta} |\nabla G_{\Omega,x}(y)|} d\sigma(y).$$

Before proving Theorem 16 we prove several intermediate results.

Lemma 18 *Define the sets S_r for $r \in (0, 1]$ by*

$$S_r = \left\{ y \in \overline{\Omega} : G_{\Omega,0}(y) = -\frac{1}{2\pi} \log(r) \right\}.$$

Then the following inequality holds true

$$|I_{\Omega}(0)|^{2-\beta} \leq \frac{r^{\beta-2}}{4\pi^2} \int_{S_r} \frac{1}{|y|^{\beta} |\nabla G_{\Omega,0}(y)|} d\sigma(y) \quad \forall r \in (0, 1].$$

Remark 19 The special case $\beta = 0$ is exactly Theorem 17 in Flucher.

Proof The set S_r is the boundary of A_r given by

$$A_r = \left\{ y \in \overline{\Omega} : G_{\Omega,0}(y) > -\frac{1}{2\pi} \log(r) \right\}.$$

Note that $0 \in A_r$ for all $r \in (0, 1]$ and

$$G_{A_r,0}(y) = G_{\Omega,0}(y) + \frac{1}{2\pi} \log(r) = -\frac{1}{2\pi} \log(|y|) - H_{A_r,0}(y),$$

where $H_{A_r,0}$ is the regular part of the Green's function $G_{A_r,0}$. We thus get

$$H_{A_r,0}(y) = H_{\Omega,0}(y) - \frac{1}{2\pi} \log(r).$$

From the definition of the conformal incenter we get

$$I_{A_r}(x) = e^{-2\pi H_{A_r,x}(x)} = r I_{\Omega}(x). \quad (11)$$

Note also that by the strong maximum principle the sets A_r are connected for all $r \in (0, 1]$. Applying Theorem 17 to the domain $\Omega = A_r$ and $\partial\Omega = S_r$ we get

$$|A_r|^{1-\frac{\beta}{2}} \leq \frac{1}{4\pi^{1+\frac{\beta}{2}}} \int_{S_r} \frac{1}{|y|^\beta |\nabla G_{\Omega,0}(y)|} d\sigma(y).$$

Now we use that $|A_r| = \pi b_r^2$ for some $b_r > 0$. It follows from Proposition 3 (d) and (11) that

$$r I_{\Omega}(0) = I_{A_r}(0) \leq b_r = \sqrt{\frac{|A_r|}{\pi}}.$$

Setting this into the previous inequality gives

$$|r I_{\Omega}(0)|^{2-\beta} \leq \left(\frac{|A_r|}{\pi} \right)^{1-\frac{\beta}{2}} \leq \frac{1}{4\pi^2} \int_{S_r} \frac{1}{|y|^\beta |\nabla G_{\Omega,0}(y)|} d\sigma(y),$$

from which the lemma follows. ■

The following lemma holds true for any domains, whether containing the origin or not. So we state this general version, although we will use it with $x = 0$.

Lemma 20 *Let $x \in \Omega$ and let $v \in W_{0,rad}^{1,2}(B_1)$. Then $P_x(v) \in W_0^{1,2}(\Omega)$ and in particular*

$$\|\nabla(P_x(v))\|_{L^2(\Omega)} = \|\nabla v\|_{L^2(B_1)}. \quad (12)$$

Moreover if $\{v_i\} \subset W_{0,rad}^{1,2}(B_1)$ concentrates at 0, then $P_x(v_i)$ concentrates at x .

Proof *Step 1.* We write $G = G_{\Omega, x}$. Let h be defined by $h(y) = e^{-2\pi G(y)}$ and hence $u(y) = v(h(y))$. In particular

$$\nabla u(y) = v'(h(y)) \nabla h(y).$$

Note that, since $G \geq 0$ in Ω we get that if $y \in h^{-1}(t) \cap \Omega$, then $t \in [0, 1]$. Thus the coarea formula gives that

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 &= \int_{\Omega} |v'(h(y))|^2 |\nabla h(y)| |\nabla h(y)| dy \\ &= \int_0^1 \left[\int_{h^{-1}(t) \cap \Omega} |v'(h(y))|^2 |\nabla h(y)| d\sigma(y) \right] dt. \end{aligned}$$

Using that $|\nabla h(y)| = 2\pi h(y) |\nabla G(y)|$, gives

$$\int_{\Omega} |\nabla u|^2 = \int_0^1 2\pi t |v'(t)|^2 \left[\int_{h^{-1}(t) \cap \Omega} |\nabla G(y)| d\sigma(y) \right] dt.$$

Note that

$$h^{-1}(t) \cap \Omega = \left\{ y \in \Omega \mid G(y) = -\frac{1}{2\pi} \log(t) \right\}.$$

Thus we obtain from Proposition 3 (b) that

$$\int_{h^{-1}(t) \cap \Omega} |\nabla G(y)| d\sigma(y) = 1 \quad \forall t \in (0, 1),$$

which implies that

$$\int_{\Omega} |\nabla u|^2 = \int_0^1 |v'(t)|^2 2\pi t dt = \int_{B_1} |\nabla v|^2.$$

This proves (12).

Step 2. Let us now assume that $\{v_i\}$ concentrates at 0 and let $\epsilon > 0$ be given. We know from Proposition 3 (e), that for some $M > 0$ big enough $\{G > M\} \subset B_{\epsilon}(x)$. Thus we obtain exactly as in Step 1 that

$$\int_{\Omega \setminus B_{\epsilon}(x)} |\nabla u_i|^2 \leq \int_{\{G \leq M\}} |\nabla u_i|^2 = \int_{e^{-2\pi M}}^1 |v'_i(t)|^2 2\pi t dt.$$

The right hand side goes to 0, since v_i concentrates. This proves that u_i concentrates too. ■

We are now able to prove the main theorem.

Proof (Theorem 16). We abbreviate again $G = G_{\Omega,0}$. From Lemma 20 we know that $u \in \mathcal{B}_1(\Omega)$. Using the coarea formula we get

$$\begin{aligned} F_{\Omega}(u) &= \int_{\Omega} \frac{(e^{\alpha u^2} - 1)}{|y|^{\beta}} \frac{|\nabla G(y)|}{|\nabla G(y)|} dy = \int_0^{\infty} \left[\int_{G^{-1}(t) \cap \Omega} \frac{(e^{\alpha u^2} - 1)}{|y|^{\beta} |\nabla G(y)|} d\sigma(y) \right] dt \\ &= \int_0^{\infty} \left(e^{\alpha v^2(e^{-2\pi t})} - 1 \right) \left[\int_{S_{r(t)}} \frac{1}{|y|^{\beta} |\nabla G(y)|} d\sigma(y) \right] dt, \end{aligned}$$

where $r(t) = e^{-2\pi t}$ and $S_{r(t)}$ is defined as in Lemma 18. That lemma therefore gives us that

$$\begin{aligned} F_{\Omega}(u) &\geq I_{\Omega}(0)^{2-\beta} \int_0^{\infty} \frac{e^{\alpha v^2(r(t))} - 1}{r(t)^{\beta}} (2\pi r(t))^2 dt \\ &= -I_{\Omega}(0)^{2-\beta} \int_0^{\infty} \frac{e^{\alpha v^2(r(t))} - 1}{r(t)^{\beta}} 2\pi r(t) r'(t) dt \\ &= I_{\Omega}(0)^{2-\beta} \int_0^1 \frac{e^{\alpha v^2(r)} - 1}{r^{\beta}} 2\pi r dr = I_{\Omega}(0)^{2-\beta} F_{B_1}(v). \end{aligned}$$

This proves the first claim of the theorem. The statement about the concentration follows directly from Lemma 20. ■

6 Domain to Ball Construction

The aim of this section is to prove the following theorem. Recall that we assume $0 \in \Omega$.

Theorem 21 (Concentration Formula) *The following formula holds*

$$F_{\Omega}^{\delta}(0) = I_{\Omega}(0)^{2-\beta} F_{B_1}^{\delta}(0).$$

The proof of this result will be a consequence of the following proposition, which allows to construct a concentrating sequence in the ball from a given concentrating sequence in Ω .

Proposition 22 *Let $\{u_i\} \subset \mathcal{B}_1(\Omega) \cap C^{\infty}(\Omega)$ be a sequence which concentrates at 0 and is a maximizing sequence for $F_{\Omega}^{\delta}(0)$. Then there exists a sequence $\{v_i\} \subset W_{0,rad}^{1,2}(B_1) \cap \mathcal{B}_1(B_1)$ concentrating at 0, such that*

$$F_{\Omega}^{\delta}(0) = \lim_{i \rightarrow \infty} F_{\Omega}(u_i) \leq I_{\Omega}^{2-\beta}(0) \liminf_{i \rightarrow \infty} F_{B_1}(v_i).$$

Proof (Theorem 21). From Lemma 4 and Proposition 22 we immediately obtain that

$$F_{\Omega}^{\delta}(0) \leq I_{\Omega}^{2-\beta}(0)F_{B_1}^{\delta}(0).$$

The reverse inequality follows from Theorem 16. ■

The proof of Proposition 22 is long and technical. We split it into many intermediate steps. To make the presentation less cumbersome, we assume in what follows that $0 \in \Omega$. However, we actually need this, and the fact that concentration occurs at 0, only in Step 6 in the proof of Lemma 29. We start with two auxiliary lemmas.

Lemma 23 *Suppose $\{u_i\} \subset \mathcal{B}_1(\Omega)$ concentrates at $x_0 \in \Omega$ and let $\{r_i\} \subset \mathbb{R}$ be such that $r_i > 0$ for all i and $\lim_{i \rightarrow \infty} r_i = 0$. Then there exists a subsequence u_{j_i} such that*

$$\lim_{i \rightarrow \infty} F_{\Omega}(u_i) = \lim_{i \rightarrow \infty} \int_{\Omega} \frac{e^{\alpha u_i^2} - 1}{|x|^{\beta}} dx = \lim_{i \rightarrow \infty} \int_{B_{2r_i}(x_0)} \frac{e^{\alpha u_{j_i}^2} - 1}{|x|^{\beta}} dx.$$

Moreover any subsequence of u_{j_i} will also satisfy the above equality.

Proof Define for each i the functions $\eta_i \in W^{1,\infty}(\mathbb{R}^2)$ by

$$\eta_i(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R}^2 \setminus B_{2r_i}(x_0) \\ \frac{|x-x_0|}{r_i} - 1 & \text{if } x \in B_{2r_i}(x_0) \setminus B_{r_i}(x_0) \\ 0 & \text{if } x \in B_{r_i}(x_0). \end{cases}$$

Note that $|\nabla \eta_i|^2 = 1/r_i^2$ in $B_{2r_i}(x_0) \setminus B_{r_i}(x_0)$. We obtain for all i, j that

$$\int_{\Omega} |\nabla(\eta_i u_j)|^2 \leq 2 \int_{\Omega} (|\nabla \eta_i|^2 |u_j|^2 + |\eta_i|^2 |\nabla u_j|^2) = A_i(u_j) + B_i(u_j),$$

where

$$A_i(u_j) = \frac{2}{r_i^2} \int_{B_{2r_i}(x_0) \setminus B_{r_i}(x_0)} |u_j|^2, \quad B_i(u_j) = 2 \int_{\Omega \setminus B_{r_i}(x_0)} |\eta_i|^2 |\nabla u_j|^2.$$

Since $|\eta_i| \leq 1$, u_j concentrates at x_0 and $u_j \rightarrow 0$ in L^2 , we get that for any fixed i the following convergences hold true

$$\lim_{j \rightarrow \infty} A_i(u_j) = 0 \quad \text{and} \quad \lim_{j \rightarrow \infty} B_i(u_j) = 0.$$

We can therefore choose a subsequence u_{j_i} such that

$$A_i(u_{j_i}) + B_i(u_{j_i}) \leq \frac{1}{2^i}.$$

We finally set $v_i = \eta_i u_{j_i} \in W_0^{1,2}(\Omega)$, which satisfies by construction $\lim_{i \rightarrow \infty} \int_{\Omega} |\nabla v_i|^2 = 0$. We obtain that $v_i \rightarrow 0$ in $W_0^{1,2}(\Omega)$. We can now apply Lemma 5 to obtain that

$$\frac{e^{\alpha v_i^2} - 1}{|x|^\beta} \rightarrow 0 \quad \text{in } L^1(\Omega).$$

In particular, recalling that $\eta_i = 1$ on $\mathbb{R}^2 \setminus B_{2r_i}(x_0)$, we get

$$0 = \lim_{i \rightarrow \infty} \int_{\Omega \setminus B_{2r_i}(x_0)} \frac{e^{\alpha v_i^2} - 1}{|x|^\beta} = \lim_{i \rightarrow \infty} \int_{\Omega \setminus B_{2r_i}(x_0)} \frac{e^{\alpha u_{j_i}^2} - 1}{|x|^\beta}.$$

From this the statement of the lemma follows immediately. ■

We will frequently use the following elementary Lemma.

Lemma 24 *Suppose $\{u_i\}$ is a sequence of measurable functions such that $u_i \rightarrow 0$ almost everywhere in Ω . Let $\{s_i\} \subset \mathbb{R}$ be a bounded sequence. Then*

$$\lim_{i \rightarrow \infty} \int_{\{u_i \leq s_i\}} \frac{e^{\alpha u_i^2} - 1}{|x|^\beta} = 0.$$

Proof Define the function f_i by

$$f_i(x) = \frac{e^{\alpha u_i^2} - 1}{|x|^\beta} \chi_i(x),$$

where χ_i is the characteristic function of the set $\{u_i \leq s_i\}$. Note that, since $\beta < 2$, we have that $|f_i| \leq g$, for some $g \in L^1(\Omega)$. Therefore by dominated convergence theorem we get that $\lim_{i \rightarrow \infty} \int_{\Omega} f_i = 0$. ■

Lemma 25 *Suppose $\{u_i\} \subset \mathcal{B}_1(\Omega) \cap C^\infty(\Omega)$ concentrates at $0 \in \Omega$ and satisfies*

$$\lim_{i \rightarrow \infty} F_{\Omega}(u_i) = F_{\Omega}^{\delta}(0). \quad (13)$$

Then for any $r > 0$ there exists $j \in \mathbb{N}$ and $k_j \in [1, 2]$ such that

$$\{u_j \geq k_j\} \cap B_r(0) \neq \emptyset \quad (14)$$

and all connected components A of $\{u_j \geq k_j\}$ will have the property:

$$\text{If } A \cap B_r(0) \neq \emptyset \quad \text{then} \quad A \subset B_{2r}(0). \quad (15)$$

Moreover A has smooth boundary.

Proof It is sufficient to prove that (14) and (15) hold with $k_j = 1$ for some $j \in \mathbb{N}$. This implies that (14) and (15) also hold for any $k \geq 1$, and hence, using Sard's theorem, one can choose $k_j \in [1, 2]$ appropriately such that A has smooth boundary in addition.

First note that for all $n \in \mathbb{N}$ there exists a $j \geq n$ such that (14) must hold. If this is not the case, then Lemma 23 and Lemma 24 imply that

$$\lim_{i \rightarrow \infty} F_{\Omega}(u_i) \leq \lim_{i \rightarrow \infty} \int_{B_r} \frac{e^{\alpha u_i^2} - 1}{|x|^{\beta}} \leq \lim_{i \rightarrow \infty} \int_{\{u_i \leq 1\}} \frac{e^{\alpha u_i^2} - 1}{|x|^{\beta}} = 0,$$

which is a contradiction to (13) (Recall that $F_{\Omega}^{\delta}(0) > 0$, see Remark 8).

Suppose now that (15) does not hold. We show that this leads to a contradiction. In that case there exists for all $j \in \mathbb{N}$ a connected component D_j of $\{u_j \geq 1\}$ and $a, b \in \Omega$ such that

$$a \in D_j \cap B_r \quad \text{and} \quad b \in D_j \cap \Omega \setminus B_{2r}.$$

For what follows we fix j and omit the explicit dependence on j (Note that a and b depend on j). Without loss of generality we can assume, by rotating the domain, that $b = (b_1, 0)$ and $b_1 \geq 2r$. Since D_j is connected, for all $x_1 \in [r, 2r]$ there exists a $x_2 \in \mathbb{R}$ such that $x = (x_1, x_2) \in D_j$. In particular $u_j(x) \geq 1$. Since Ω is bounded, there exists an $M > 0$, which is independent of the rotation of the domain (and hence of j), such that

$$\Omega \subset [-M, M] \times [-M, M] = [-M, M]^2.$$

Let us extend u_j by zero in $[-M, M]^2 \setminus \Omega$. We obtain in this way (using Hölder inequality in the last inequality) that for any $x_1 \in [r, 2r]$

$$\begin{aligned} 1 \leq u_j(x) &= u_j(x_1, x_2) - u_j(x_1, -M) = \int_{-M}^{x_2} \frac{\partial u_j}{\partial x_2}(x_1, s) ds \\ &\leq \int_{-M}^M \left| \frac{\partial u_j}{\partial x_2}(x_1, s) \right| ds \leq \sqrt{2M} \left(\int_{-M}^M |\nabla u_j(x_1, s)|^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Taking the square of the previous inequality and integrating x_1 from r to $2r$ gives

$$r \leq 2M \int_r^{2r} \int_{-M}^M |\nabla u_j(x_1, s)|^2 ds dx_1 \leq 2M \int_{\Omega \setminus B_r} |\nabla u_j|^2.$$

But this cannot hold true for all j , since u_j concentrates at 0. ■

The next lemma is about the first modification of the the sequence $\{u_i\}$ given in Proposition 22.

Lemma 26 *Let $\{u_i\} \subset \mathcal{B}_1(\Omega) \cap C^\infty(\Omega)$ be a sequence which concentrates at $0 \in \Omega$ and satisfies*

$$\lim_{i \rightarrow \infty} F_\Omega(u_i) = F_\Omega^\delta(0).$$

Then there exists a sequence $\{v_i\} \subset \mathcal{B}_1(\Omega)$ and sequences $r_i > 0$, with $r_i \rightarrow 0$ and $\{k_i\} \in [1, 2]$ such that

$$\{v_i \geq k_i\} \subset B_{2r_i}, \quad \Delta v_i = 0 \quad \text{in } \{v_i < k_i\}.$$

Moreover v_i has the properties: there exist a sequence $\{\lambda_i\} \subset \mathbb{R}$, $\lambda_i > 0$ such that

- (i) $\lim_{i \rightarrow \infty} \lambda_i = \infty$
- (ii) $\lim_{i \rightarrow \infty} v_i(y) = 0$ for all y in $\Omega \setminus \{0\}$
- (iii) $\lambda_i v_i \rightarrow G_{\Omega,0}$ in $C_{loc}^2(\Omega \setminus \{0\})$
- (iv) $\lim_{i \rightarrow \infty} F_\Omega(v_i) = F_\Omega^\delta(0).$

Proof *Step 1.* Take a sequence of positive real numbers r_i such that $\lim_{i \rightarrow \infty} r_i = 0$ and choose a subsequence of u_i , using Lemma 23, such that

$$F_\Omega^\delta(0) = \lim_{i \rightarrow \infty} F_\Omega(u_i) = \lim_{i \rightarrow \infty} \int_{B_{r_i}} \frac{e^{\alpha u_i^2} - 1}{|x|^\beta}. \quad (16)$$

Choosing again a subsequence we can assume by Lemma 25 that there exist $k_i \in [1, 2]$ such that all connected components A of $\{u_i \geq k_i\}$ which intersect B_{r_i} are contained in B_{2r_i} . We define A_i as the union of all such A . We also know from Lemma 25 that A_i is not empty. Let w_i be the solution of (this exists because A_i has smooth boundary)

$$\begin{cases} \Delta w_i = 0 & \text{in } \Omega \setminus \overline{A_i} \\ w_i = 0 & \text{on } \partial\Omega, \quad w_i = k_i & \text{on } \partial A_i. \end{cases}$$

We now define $\overline{u}_i \in W_0^{1,2}(\Omega)$ as

$$\overline{u}_i = \begin{cases} u_i & \text{in } A_i \\ w_i & \text{in } \Omega \setminus A_i. \end{cases}$$

Since harmonic functions minimize the Dirichlet integral we have $\|\nabla \overline{u}_i\|_{L^2} \leq \|\nabla u_i\|_{L^2}$. Thus we have constructed a sequence which has the properties:

$$\{\overline{u}_i \geq k_i\} \subset B_{2r_i}, \quad \Delta \overline{u}_i = 0 \quad \text{in } \{\overline{u}_i < k_i\} \quad \text{and} \quad \|\nabla \overline{u}_i\|_{L^2} \leq 1.$$

Step 2. We will show in this Step that for all $y \in \Omega \setminus \{0\}$ we have $\overline{u}_i(y) > 0$ for all i large enough and $\lim_{i \rightarrow \infty} \overline{u}_i(y) = 0$. The fact that $\overline{u}_i(y) > 0$ follows from the

maximum principle. Since Ω is bounded there exists $M > 0$ such that $\overline{\Omega} \subset B_M$. Define $W_i = B_M \setminus \overline{B_{2r_i}}$ and let ψ_i be the solution of

$$\begin{cases} \Delta \psi_i = 0 & \text{in } W_i \\ \psi_i = 2 & \text{on } \partial B_{2r_i} \quad \text{and} \quad \psi_i = 0 & \text{on } \partial B_M. \end{cases}$$

The function ψ_i can be given explicitly:

$$\psi_i = \frac{2}{\log\left(\frac{2r_i}{M}\right)} \log\left(\frac{|x|}{M}\right).$$

Recall that $k_i \in [1, 2]$ and note that

$$\begin{aligned} \psi_i &> 0 \quad \text{and} \quad \overline{u}_i = 0 && \text{on } \partial\Omega, \\ \psi_i &= 2 \quad \text{and} \quad \overline{u}_i < k_i \leq 2 && \text{on } \partial B_{2r_i}, \end{aligned}$$

and thus $\psi_i - \overline{u}_i > 0$ on ∂W_i . Since \overline{u}_i is also harmonic in W_i the maximum principle implies that $\overline{u}_i \leq \psi_i$ in W_i . For i big enough $y \in W_i$ and the claim of Step 2 follows from the fact that $\lim_{i \rightarrow \infty} \psi_i(y) = 0$.

Step 3. Choose $y \in \Omega \setminus \{0\}$ and define λ_i by

$$\lambda_i = \frac{G_{\Omega,0}(y)}{\overline{u}_i(y)} \quad \Leftrightarrow \quad \lambda_i \overline{u}_i(y) = G_{\Omega,0}(y) \quad (17)$$

In view of Step 2 this is well defined, $\lambda_i > 0$ and

$$\lim_{i \rightarrow \infty} \lambda_i = \infty.$$

Let $y \in K_1 \subset \Omega \setminus \{0\}$ be a compact set. Choose another compact set K_2 , such that $K_1 \subset\subset K_2 \subset \Omega \setminus \{0\}$. Applying Harnack inequality on K_2 we get that there exist $c_1, c_2 > 0$, such that

$$c_1 |G_{\Omega,0}(y)| \leq |\lambda_i \overline{u}_i(x)| \leq c_2 |G_{\Omega,0}(y)| \quad \forall x \in K_2 \quad \text{and} \quad \forall i \text{ large enough.}$$

Hence the sequence $\lambda_i \overline{u}_i$ is uniformly bounded in the $C^0(K_2)$ norm. Choose $0 < \alpha < 1$. It follows from Schauder estimates (see Gilbarg-Trudinger, Corollary 6.3 page 93 and the remark thereafter) that $\lambda_i \overline{u}_i$ is also uniformly bounded in the $C^{2,\alpha}(K_1)$ norm. Using the compact embedding $C^{2,\alpha}(K_1) \hookrightarrow C^2(K_1)$ we obtain that there exists $g \in C^2(K_1)$ and a subsequence \overline{u}_i with

$$\lambda_i \overline{u}_i \rightarrow g \quad \text{in } C^2(K_1).$$

We finally define $v_i = \overline{u}_i$ as this subsequence. It follows from (17) and Bocher's theorem (see for instance [2] Theorem 3.9 page 50) that $g = G_{\Omega,0}$.

Step 4. It remains to prove (iv). Recall that $\bar{u}_i \leq k_i$ in $\Omega \setminus A_i$. We therefore obtain, using Lemma 24 twice and the definition of A_i that

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{\Omega} \frac{e^{\alpha \bar{u}_i^2} - 1}{|x|^\beta} &= \lim_{i \rightarrow \infty} \int_{A_i} \frac{e^{\alpha u_i^2} - 1}{|x|^\beta} \geq \lim_{i \rightarrow \infty} \int_{A_i \cap B_{r_i}} \frac{e^{\alpha u_i^2} - 1}{|x|^\beta} \\ &= \lim_{i \rightarrow \infty} \int_{\{u_i \geq k_i\} \cap B_{r_i}} \frac{e^{\alpha u_i^2} - 1}{|x|^\beta} = \lim_{i \rightarrow \infty} \int_{B_{r_i}} \frac{e^{\alpha u_i^2} - 1}{|x|^\beta} = F_{\Omega}^{\delta}(0), \end{aligned}$$

where we have used (16) in the last equality. ■

The next lemma is about the second modification of the sequence $\{u_i\}$ given in Proposition 22, following the first modification given by Lemma 26.

Lemma 27 *Let $\{u_i\} \subset W_0^{1,2}(\Omega)$ be a sequence and λ_i a sequence in \mathbb{R} such that $\lambda_i \rightarrow \infty$,*

$$\lambda_i u_i \rightarrow G_{\Omega,0} \quad \text{in } C_{loc}^0(\Omega \setminus \{0\}) \quad \text{and} \quad \Delta u_i = 0 \text{ in } \{u_i < 1\}.$$

Then there exists a subsequence λ_{i_l} and a sequence $\{v_l\} \subset W_0^{1,2}(\Omega)$ such that the following properties hold true:

- (a) $\lambda_{i_l} \geq l$
- (b) *The sets $\{v_l \geq l/\lambda_{i_l}\}$ are approximately small disks at 0 as $l \rightarrow \infty$.*
- (c) $v_l(x) \rightarrow 0$ as $l \rightarrow \infty$ for every x in $\Omega \setminus \{0\}$.
- (d) *For every l*

$$\int_{\Omega} |\nabla v_l|^2 \leq \int_{\Omega} |\nabla u_{i_l}|^2.$$

- (e) *The inequality $v_l \geq u_{i_l}$ holds in Ω . In particular $F_{\Omega}(v_l) \geq F_{\Omega}(u_{i_l})$.*

Remark 28 (i) Flucher in his paper [5] (see Point 4 page 492) claims that the hypothesis $\lambda_i u_i \rightarrow G_{\Omega, x_0}$ in $C_{loc}^1(\Omega \setminus \{0\})$ implies that for some subsequence the sets $\{u_i \geq 1\}$ form approximately small disks. Lemma 27 actually shows that to obtain this property it is not sufficient to chose a subsequence, but the sequence has to be modified again. This is necessary, as shown by the following example: let $\Omega = B_1(0)$, $\lambda_i = i$ for all i and define

$$u_i(x) = \begin{cases} -\frac{1}{2\pi i} \log(|x|) & \text{if } |x| \geq e^{-i\pi} \\ \frac{1}{2} & \text{if } |x| \leq e^{-i\pi}. \end{cases}$$

Obviously $\lambda_i u_i \rightarrow G_{B_1,0}$ in $C_{loc}^{\infty}(\Omega \setminus \{0\})$. But the sets $\{u_i \geq 1\}$ are empty for all i . One can easily construct an example where even $u_i(0) = 0$ for all i and not even the sets $\{u_i \geq s_i\}$ will have the desired property, for any sequence $s_i > 0$.

Proof Again we abbreviate $G = G_{\Omega,0}$.

Step 1. Let $l \in \mathbb{N}$. We know from Proposition 3 (e) that the sets $\{G \geq l\}$ form approximately small disks, that is $B_{\rho_l - \delta_l} \subset \{G \geq l\} \subset B_{\rho_l + \delta_l}$, for two sequences ρ_l and δ_l tending to zero and satisfying $\lim_{l \rightarrow \infty} (\delta_l / \rho_l) = 0$. We know from maximum principle for harmonic functions, respectively Hopf lemma that the following strict inequalities hold

$$G < l \quad \text{in } \overline{\Omega} \setminus B_{\rho_l + 2\delta_l} \quad \text{and} \quad G > l \quad \text{on } \partial B_{\rho_l - 2\delta_l}.$$

Let $\epsilon_1 > 0$ be such that $G \leq l - \epsilon_1$ on $\overline{\Omega} \setminus B_{\rho_l + 2\delta_l}$. Using now the locally uniform convergence of the hypothesis, we know that there exists $j_l \in \mathbb{N}$ such that

$$\|\lambda_i u_i - G\|_{C^0(\overline{\Omega}_\eta \setminus B_{\rho_l + 2\delta_l})} \leq \epsilon_1 \quad \forall i \geq j_l,$$

where $\eta > 0$ and $\Omega_\eta = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \eta\}$ is choosen such that $\overline{B}_{\rho_l + 2\delta_l} \in \Omega_\eta$ for all l . In particular this implies that

$$\lambda_i u_i \leq l \quad \text{in } \overline{\Omega}_\eta \setminus B_{\rho_l + 2\delta_l} \quad \forall i \geq j_l. \quad (18)$$

Let $\epsilon_2 > 0$ be such that $G \geq l + \epsilon_2$ on $\partial B_{\rho_l - 2\delta_l}$. We use again locally uniform convergence and choose $i_l \geq j_l$ such that

$$\|\lambda_{i_l} u_{i_l} - G\|_{C^0(\partial B_{\rho_l - 2\delta_l})} \leq \epsilon_2.$$

Moreover, since $\lambda_i \rightarrow \infty$ we can assume, by choosing i_l if necessary even larger, that

$$\lambda_{i_l} \geq l. \quad (19)$$

In particular we obtain that

$$\lambda_{i_l} u_{i_l} \geq l \quad \text{on } \partial B_{\rho_l - 2\delta_l}. \quad (20)$$

Finally we define the set A_l by

$$A_l = \{x \in B_{\rho_l - 2\delta_l} : \lambda_{i_l} u_{i_l} < l\}. \quad (21)$$

At last we define

$$v_l = \begin{cases} u_{i_l} & \text{in } \overline{\Omega} \setminus A_l \\ \frac{l}{\lambda_{i_l}} & \text{in } A_l. \end{cases} \quad (22)$$

Note that $v_l \in W_0^{1,2}(\Omega)$ in view of (20) and (21).

Step 2. Let us now verify that the new sequence v_l verifies (a)–(e). The first statement (a) is satisfied obviously by (19). Using the hypothesis $\Delta u_i = 0$ in $\{u_i < 1\}$

and again (19) we get that $\Delta u_{i_l} = 0$ in $\{u_{i_l} < l/\lambda_{i_l}\}$. Thus from the maximum principle we also have

$$\lambda_{i_l} u_{i_l} \leq l \quad \text{in } \overline{\Omega} \setminus \Omega_\eta.$$

Together with (18) we get

$$\{\lambda_{i_l} u_{i_l} \geq l\} \subset B_{\rho_l + 2\delta_l},$$

which implies, by the definition of v_l , that $\{v_l \geq l/\lambda_{i_l}\} \subset B_{\rho_l + 2\delta_l}$. Moreover from the definition of A_l and the definition of v_l we get that $B_{\rho_l - 2\delta_l} \subset \{v_l \geq l/\lambda_{i_l}\}$. This shows (b) indeed:

$$B_{\rho_l - 2\delta_l} \subset \left\{v_l \geq \frac{l}{\lambda_{i_l}}\right\} \subset B_{\rho_l + 2\delta_l}.$$

Let us now show (c). Let $x \in \Omega \setminus \{0\}$. Then for all l big enough we get that $x \in \Omega \setminus B_{\rho_l + 2\delta_l}$. So for those l we have $v_l = u_{i_l}$. Since we have that $\lambda_{i_l} \rightarrow \infty$ and that

$$\lambda_{i_l} u_{i_l}(x) \rightarrow G(x),$$

we must have that $u_{i_l}(x) \rightarrow 0$, which proves (c). The statement (d) follows immediately from the definition (21) of v_l . (e) follows also directly from (21) and (22). \blacksquare

After having modified the sequence $\{u_i\}$ given in Proposition 22 in the two previous lemmas, we finally construct the appropriate corresponding sequence $\{v_i\} \subset W_{0,rad}^{1,2}(B_1)$. This is contained in the following lemma.

Lemma 29 *Let $\{u_i\} \subset W_0^{1,2}(\Omega)$ and $\{s_i\} \subset \mathbb{R}$ be sequences with the following properties:*

$$s_i \leq 1 \quad \forall i \in \mathbb{N},$$

the sets $\{u_i \geq s_i\}$ are approximately small disks at 0 as $i \rightarrow \infty$ and moreover suppose that pointwise $u_i(x) \rightarrow 0$ for all $x \in \Omega \setminus \{0\}$. Then there exists a sequence $\{v_i\} \subset W_{0,rad}^{1,2}(B_1)$ such that for all i

$$\|\nabla v_i\|_{L^2(B_1)} \leq \|\nabla u_i\|_{L^2(\Omega)}$$

and, assuming that the left hand side limit exists,

$$\lim_{i \rightarrow \infty} F_\Omega(u_i) \leq I_\Omega(0)^{2-\beta} \liminf_{i \rightarrow \infty} F_{B_1}(v_i).$$

Moreover $v_i(x) \rightarrow 0$ for all $x \in B_1 \setminus \{0\}$ and if v_i concentrates at some $x_0 \in B_1$, then $x_0 = 0$.

Remark 30 (i) Flucher in his paper [5] states and proves this result only for the constant sequence $s_i = 1$ for all i . This is not sufficient to prove Proposition 22, not

even for the Moser-Trudinger functional (i.e. $\beta = 0$). However, in Flucher's thesis [6], this proof is correct. Unfortunately, this thesis is not easily accessible.

(ii) Note that we make no assumption on the radius of the approximately small disks $\{u_i \geq s_i\}$, nor do we assume any kind of convergence of the u_i towards $G_{\Omega,0}$.

Proof Throughout this proof $G = G_{\Omega,0}$ shall denote the Green's function of Ω with singularity at 0. Recall that by assumption there exists real positive numbers ρ_i and ϵ_i such that for $i \rightarrow \infty$

$$\rho_i \rightarrow 0 \quad \text{and} \quad \frac{\epsilon_i}{\rho_i} \rightarrow 0, \quad (23)$$

satisfying for all i the following inclusion

$$B_{\rho_i - \epsilon_i} \subset \{u_i \geq s_i\} \subset B_{\rho_i + \epsilon_i}. \quad (24)$$

Step 1. Let us define λ_i , implicitly, by the following equation:

$$\rho_i = I_{\Omega}(0)e^{-2\pi\lambda_i}, \quad (25)$$

that is

$$\lambda_i = -\frac{1}{2\pi} \log \left(\frac{\rho_i}{I_{\Omega}(0)} \right).$$

Note that $\lambda_i \rightarrow \infty$ as $i \rightarrow \infty$. We claim that there exists $t_i \geq \lambda_i$ such that

$$\lim_{i \rightarrow \infty} (t_i - \lambda_i) = 0 \quad (26)$$

and

$$\{G \geq t_i\} \subset \{u_i \geq s_i\}. \quad (27)$$

We know from Proposition 3 (e) that if $t_i \geq 0$ is such that $t_i \rightarrow \infty$, then there exists $\sigma_i > 0$ such that

$$\lim_{i \rightarrow \infty} \frac{\sigma_i}{\tau_i} = 0 \quad \text{and} \quad B_{\tau_i - \sigma_i} \subset \{G \geq t_i\} \subset B_{\tau_i + \sigma_i},$$

where $\tau_i = I_{\Omega}(0)e^{-2\pi t_i}$. In view of (24) it is therefore sufficient to choose t_i such that

$$\tau_i + \sigma_i = \rho_i - \epsilon_i. \quad (28)$$

It remains to show that with this choice (26) is also satisfied. Using (25) and solving the previous equation for t_i explicitly gives that

$$t_i = \lambda_i - \frac{1}{2\pi} \log \left(1 - \frac{\epsilon_i + \sigma_i}{\rho_i} \right).$$

Since we know from (23) that $\epsilon_i/\rho_i \rightarrow 0$, it is sufficient to show that $\sigma_i/\rho_i \rightarrow 0$. We obtain from (28) that

$$\frac{\sigma_i}{\tau_i} = \frac{\sigma_i}{\rho_i - \epsilon_i - \sigma_i} = \frac{\sigma_i}{\rho_i \left(1 - \frac{\epsilon_i}{\rho_i} - \frac{\sigma_i}{\rho_i}\right)}.$$

Solving this equation for (σ_i/ρ_i) and using that $\epsilon_i/\rho_i \rightarrow 0$ and $\sigma_i/\tau_i \rightarrow 0$ shows that also $(\sigma_i/\rho_i) \rightarrow 0$. This proves (26).

Step 2. In this step we will show that

$$\int_{\{u_i < s_i\}} |\nabla u_i|^2 \geq \frac{s_i^2}{t_i}. \quad (29)$$

Let us denote

$$U = \{u_i \geq s_i\} \quad \text{and} \quad V = \{G \geq t_i\}.$$

From Step 1 we know that $V \subset U$ and by assumption $U \subset \Omega$. Let h_i be the unique solution of the problem

$$\begin{cases} \Delta h_i = 0 & \text{in } \Omega \setminus V \\ h_i = 0 & \text{on } \partial\Omega \quad \text{and} \quad h_i = 1 \quad \text{on } \partial V. \end{cases}$$

We see that this is satisfied precisely by $h_i = G/t_i$. Let us define $w_i \in W^{1,2}(\Omega \setminus V)$ by

$$w_i = \begin{cases} \frac{u_i}{s_i} & \text{in } \Omega \setminus U \\ 1 & \text{in } U \setminus V. \end{cases}$$

Note that w_i has the same boundary values as h_i on the boundary of $\Omega \setminus V$. Since h_i minimizes the Dirichlet integral among all such functions we get that

$$\int_{\Omega \setminus V} |\nabla h_i|^2 \leq \int_{\Omega \setminus V} |\nabla w_i|^2 = \int_{\Omega \setminus U} |\nabla w_i|^2 = \frac{1}{s_i^2} \int_{\{u_i < s_i\}} |\nabla u_i|^2.$$

From Proposition 3 (a) we know that

$$\int_{\Omega \setminus V} |\nabla h_i|^2 = \int_{\{G < t_i\}} \left| \nabla \left(\frac{G}{t_i} \right) \right|^2 = \frac{1}{t_i}.$$

Setting this into the previous inequality proves (29).

Step 3. In this step we will define $v_i \in W_{0,rad}^{1,2}(B_1)$. Let $\Omega^* = B_R$ be the symmetrized domain and $u_i^* \in W_{0,rad}^{1,2}(B_R)$ be the radially decreasing symmetric rearrangement of u_i . Then there exists $0 < a_i < R$ such that

$$\{u_i^* \geq s_i\} = B_{a_i}. \quad (30)$$

Moreover define $0 < \delta_i < 1$ by $\delta_i = e^{-2\pi t_i}$. At last we can define v_i as

$$v_i(x) = \begin{cases} -\frac{s_i}{2\pi t_i} \log(x) & \text{if } x \geq \delta_i \\ u_i^* \left(\frac{a_i}{\delta_i} x \right) & \text{if } x \leq \delta_i. \end{cases}$$

Note that v_i belongs indeed to $W^{1,2}(B_1)$ since the two values coincide if $x = \delta_i$.

Step 4. In this Step we will show that $\|\nabla v_i\|_{L^2(B_1)} \leq \|\nabla u_i\|_{L^2(\Omega)}$. Let us denote

$$A_i = \int_{B_1 \setminus B_{\delta_i}} |\nabla v_i|^2 \quad \text{and} \quad D_i = \int_{B_{\delta_i}} |\nabla v_i|^2.$$

A direct calculation gives that

$$A_i = s_i^2 \int_{\delta_i}^1 \frac{2\pi}{(2\pi t_i)^2 r} dr = \frac{s_i^2}{t_i}.$$

Using a change of variables and Proposition 2 (ii) gives that

$$D_i = \int_{B_{a_i}} |\nabla u_i^*|^2 = \int_{\{u_i^* \geq s_i\}} |\nabla u_i^*|^2 \leq \int_{\{u_i \geq s_i\}} |\nabla u_i|^2.$$

Finally we get that, using (29), that

$$\int_{B_1} |\nabla v_i|^2 = D_i + A_i \leq \int_{\Omega} |\nabla u_i|^2 - \int_{\{u_i < s_i\}} |\nabla u_i|^2 + \frac{s_i^2}{t_i} \leq \int_{\Omega} |\nabla u_i|^2.$$

Step 5. In this step we show that

$$\lim_{i \rightarrow \infty} \frac{a_i}{\delta_i} = I_{\Omega}(0).$$

Using the fact that $|\{u_i^* \geq s_i\}| = |\{u_i \geq s_i\}|$, (30) and the hypothesis (24) we obtain the inequality $\rho_i - \epsilon_i \leq a_i \leq \rho_i + \epsilon_i$. From this we obtain that

$$\frac{\rho_i}{\delta_i} \left(1 - \frac{\epsilon_i}{\rho_i} \right) \leq \frac{a_i}{\delta_i} \leq \frac{\rho_i}{\delta_i} \left(1 + \frac{\epsilon_i}{\rho_i} \right).$$

From the hypothesis (23) we know that $\epsilon_i/\rho_i \rightarrow 0$. It is therefore sufficient to calculate the limit of ρ_i/δ_i . In view the definition of ρ_i and (26) this is indeed equal to

$$\lim_{i \rightarrow \infty} \frac{\rho_i}{\delta_i} = \lim_{i \rightarrow \infty} I_{\Omega}(0) e^{2\pi(t_i - \lambda_i)} = I_{\Omega}(0),$$

which proves the statement of this step.

Step 6 (equality of functional limit). Let us first show that both u_i and v_i converge to zero almost everywhere. For u_i this holds true by hypothesis. So let $x \in B_1 \setminus \{0\}$ be given and note that for all i big enough

$$x \geq e^{-2\pi\sqrt{t_i}} \geq e^{-2\pi t_i} = \delta_i.$$

Therefore we obtain from the definition of v_i and the fact that s_i are bounded, that

$$v_i(x) \leq -\frac{s_i}{2\pi t_i} \log \left(e^{-2\pi\sqrt{t_i}} \right) = \frac{s_i}{\sqrt{t_i}} \rightarrow 0,$$

which shows the claim also for v_i . In view of Lemma 24 it is therefore sufficient to show that

$$\lim_{i \rightarrow \infty} \int_{\{u_i \geq s_i\}} \frac{e^{\alpha u_i^2} - 1}{|x|^\beta} = I_\Omega^{2-\beta}(0) \lim_{i \rightarrow \infty} \int_{\{v_i \geq s_i\}} \frac{e^{\alpha v_i^2} - 1}{|x|^\beta}. \quad (31)$$

From Proposition 2 (i) and the properties of symmetrization we get that for every i

$$\int_{\{u_i \geq s_i\}} \frac{e^{\alpha u_i^2} - 1}{|x|^\beta} \leq \int_{\{u_i^* \geq s_i\}} \frac{e^{\alpha (u_i^*)^2} - 1}{(|x|^\beta)^*} = \int_{B_{a_i}} \frac{e^{\alpha (u_i^*)^2} - 1}{(|x|^\beta)^*}.$$

(If $\beta = 0$ then the inequality can actually be replaced by an equality, see Kesavan, page 14, equation (1.3.2)). For i big enough $B_{a_i} \subset \Omega$, and then $(|x|^\beta)^* = |x|^\beta$ for all $x \in B_{a_i}$. Making the substitution $x = (a_i/\delta_i) y$ gives

$$\int_{\{u_i \geq s_i\}} \frac{e^{\alpha u_i^2} - 1}{|x|^\beta} \leq \left(\frac{a_i}{\delta_i} \right)^{2-\beta} \int_{B_{\delta_i}} \frac{e^{\alpha v_i^2} - 1}{|y|^\beta} = \left(\frac{a_i}{\delta_i} \right)^{2-\beta} \int_{\{v_i \geq s_i\}} \frac{e^{\alpha v_i^2} - 1}{|x|^\beta}.$$

From Step 5 we therefore get that

$$\lim_{i \rightarrow \infty} \int_{\{u_i \geq s_i\}} \frac{e^{\alpha u_i^2} - 1}{|x|^\beta} \leq I_\Omega^{2-\beta}(0) \liminf_{i \rightarrow \infty} \int_{\{v_i \geq s_i\}} \frac{e^{\alpha v_i^2} - 1}{|x|^\beta},$$

which proves (31) and hence concludes the proof of the lemma. ■

We are now able to prove the main proposition of this section.

Proof (Proposition 22). We know from Lemma 26 that there exists a sequence, which we call again $\{u_i\} \subset \mathcal{B}_1(\Omega)$, and a sequence λ_i such that the properties (i)–(iv) of Lemma 26 are satisfied. We now apply Lemma 27 to get a new sequence $\{v_l\} \subset \mathcal{B}_1(\Omega)$ which satisfies properties (a)–(d). Moreover we obtain from property (iv) of u_i and (e) that

$$\liminf_{l \rightarrow \infty} F_\Omega(v_l) \geq F_\Omega^\delta(0).$$

Let us again rename λ_{i_l} by λ_i and v_l by u_i . We define

$$s_i = \frac{i}{\lambda_i}.$$

By (a) we obtain that $s_i \leq 1$ for all i . By (b) the hypothesis $\{u_i \geq s_i\}$ being approximately small disks of Lemma 29 is satisfied. We therefore obtain from Lemma 29 (taking again a subsequence which achieves \liminf) that there exists $\{v_i\} \subset W_{0,rad}^{1,2}(B_1) \cap \mathcal{B}_1(B_1)$ such that

$$F_\Omega^\delta(0) \leq \lim_{i \rightarrow \infty} F_\Omega(u_i) \leq I_\Omega^{2-\beta}(0) \liminf_{i \rightarrow \infty} F_{B_1}(v_i).$$

It remains to show that the $\{v_i\}$ has to concentrate at 0. If v_i does not concentrate at 0, then (cf. Lemma 29) it does not concentrate at all. We therefore get from the concentration compactness alternative Theorem 6 that for some subsequence

$$\liminf_{i \rightarrow \infty} F_{B_1}(v_i) = \lim_{i \rightarrow \infty} F_{B_1}(v_i) = 0.$$

But this leads to the contradiction $F_\Omega^\delta(0) = 0$, see Remark 8. ■

7 Proof of the Main Theorem

We now prove Theorem 1.

Proof In view of Proposition 9 we can assume that $0 \in \overline{\Omega}$. We distinguish to cases.

Case 1: $0 \in \Omega$. From Theorems 21, 15 and 16 we know that

$$F_\Omega^\delta(0) = I_\Omega^{2-\beta}(0) F_{B_1}^\delta(0) < I_\Omega^{2-\beta}(0) F_{B_1}^{\sup} \leq F_\Omega^{\sup}.$$

Thus we obtain, using also Proposition 7, that $F_\Omega^\delta(x) < F_\Omega^{\sup}$ for all $x \in \overline{\Omega}$. This implies that maximizing sequences cannot concentrate and the result follows from Theorem 6.

Case 2: $0 \in \partial\Omega$. We will show that $F_\Omega^\delta(0) = 0$ in this case. Let $\{u_i\} \subset \mathcal{B}_1(\Omega)$ be a sequence concentrating at 0 such that

$$\lim_{i \rightarrow \infty} F_\Omega(u_i) = F_\Omega^\delta(0).$$

Choose a sequence of bounded smooth open domains Ω_n which have the property

$$0 \in \Omega_n, \quad \Omega \subset \Omega_n \quad \forall n \in \mathbb{N} \quad \text{and} \quad \lim_{n \rightarrow \infty} |0 - \partial\Omega_n| = 0,$$

where $|0 - \partial\Omega_n|$ denotes the distance between 0 and $\partial\Omega_n$. Define for each $n \in \mathbb{N}$ the functions $u_i^n \in \mathcal{B}_1(\Omega_n)$ by extending u_i by zero in $\Omega_n \setminus \overline{\Omega}$. Note that for each fixed n

the sequence $\{u_i^n\}_{i \in \mathbb{N}}$ concentrates at 0. We therefore obtain from Theorem 21 that for each n

$$\lim_{i \rightarrow \infty} F_{\Omega}(u_i) = \lim_{i \rightarrow \infty} F_{\Omega_n}(u_i^n) \leq F_{\Omega_n}^{\delta}(0) = I_{\Omega_n}^{2-\beta}(0)F_{B_1}^{\delta}(0).$$

Thus we have shown that for every n

$$F_{\Omega}^{\delta}(0) \leq I_{\Omega_n}^{2-\beta}(0)F_{B_1}^{\delta}(0).$$

We now let $n \rightarrow \infty$ and use the estimate (see Flucher [5] page 485, proof of Proposition 12 Part 2)

$$I_{\Omega_n}(0) \leq 6|0 - \partial\Omega_n|,$$

to obtain that $F_{\Omega}^{\delta}(0) = 0$. So if $0 \in \partial\Omega$, then $F_{\Omega}^{\delta}(x) = 0$ for all $x \in \overline{\Omega}$. We conclude as in Case 1. ■

Acknowledgements We have benefitted from helpful discussions with A. Adimurthi, K. Sandeep and M. Struwe.

References

- [1] Adimurthi A. and Sandeep K., A singular Moser-Trudinger embedding and its applications, *NoDEA Nonlinear Differential Equations Appl.*, **13** (2007), no. 5-6, 585–603.
- [2] Axler S., Bourdon P. and Ramey W., *Harmonic function theory*, Second edition, Graduate Texts in Mathematics, 137, Springer-Verlag, New York, 2001.
- [3] Carleson L. and Chang S.-Y. A., On the existence of an extremal function for an inequality by J. Moser, *Bull. Sci. Math.*, (2) 110 (1986), no. 2, 113–127.
- [4] Csátó G., An isoperimetric problem with density and the Hardy Sobolev inequality in \mathbb{R}^2 , preprint. <http://arxiv.org/abs/1410.8041>
- [5] Flucher M., Extremal functions for the Trudinger-Moser inequality in 2 dimensions, *Comment. Math. Helvetici*, **67** (1992), 471–497.
- [6] Flucher M., Extremal functions for the Trudinger-Moser inequality in 2 dimensions, Ph.D. thesis, ETH Zürich, 1991.
- [7] Kesavan S., *Symmetrization and applications*, Series in Analysis, 3. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.
- [8] Lions P.-L., The concentration-compactness principle in the calculus of variations. The limit case. I, *Rev. Mat. Iberoamericana* **1**, (1985), no. 1, 145–201.
- [9] Malchiodi A. and Martinazzi L., Critical points of the Moser-Trudinger functional on a disk, *J. Eur. Math. Soc. (JEMS)*, **16** (2014), no. 5, 893–908.
- [10] Moser J., A sharp form of an inequality by N. Trudinger, *Indiana Univ. Math. J.*, **20**, (1971), no. 11, 1077–1092.
- [11] Struwe M., Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **5** (1988), no. 5, 425–464.
- [12] Trudinger N.S., On embeddings into Orlicz spaces and some applications, *J. Math. Mech.*, **17** (1967), 473–484.